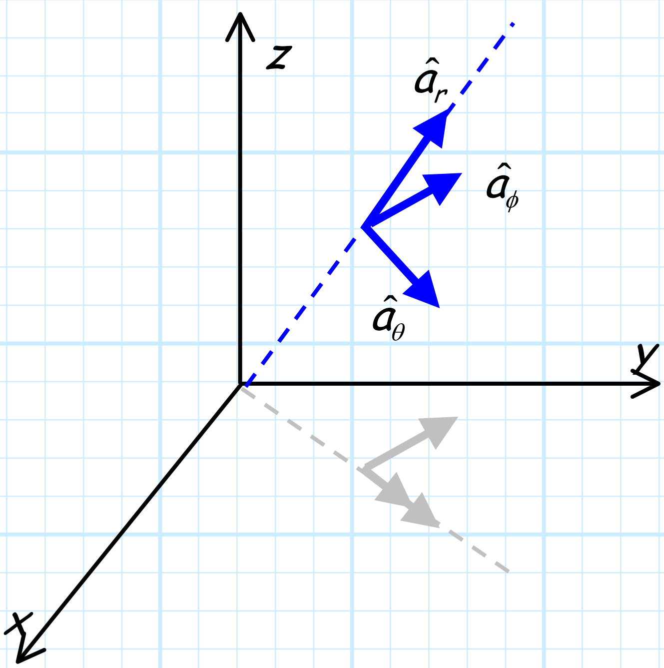
Spherical Base Vectors

Spherical base vectors are the "natural" base vectors of a **sphere**.

$\hat{a}_r$ points in the direction of **increasing** r. In other words $\hat{a}_r$ points **away from the origin**. This is analogous to the direction we call **up**.

$\hat{a}_\theta$ points in the direction of **increasing** θ. This is analogous to the direction we call **south.**

$\hat{a}_\phi$ points in the direction of increasing ϕ. This is analogous to the direction we call **east.**



IMPORTANT NOTE: The directions of spherical base vectors are **dependent on position**. First you must determine **where** you are in space (using coordinate values), **then** you can define the directions of $\hat{a}_r$, $\hat{a}_\theta$, $\hat{a}_\phi$.

Note **Cartesian** base vectors are **special**, in that their directions are **independent** of location—they have the same directions throughout all space.

Thus, it is helpful to define spherical base vectors **in terms of** Cartesian base vectors. It can be shown that:

$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos\phi \qquad \hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos\phi \qquad \hat{a}_\phi \cdot \hat{a}_x = -\sin\phi$$

$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin\phi \qquad \hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin\phi \qquad \hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

$$\hat{a}_r \cdot \hat{a}_z = \cos\theta \qquad \hat{a}_\theta \cdot \hat{a}_z = -\sin\theta \qquad \hat{a}_\phi \cdot \hat{a}_z = 0$$

Recall that **any** vector **A** can be written as:

$$\mathbf{A} = \left(\mathbf{A}\cdot\hat{a}_x\right)\hat{a}_x + \left(\mathbf{A}\cdot\hat{a}_y\right)\hat{a}_y + \left(\mathbf{A}\cdot\hat{a}_z\right)\hat{a}_z .$$

Therefore, we can write unit vector $\hat{a}_r$ as, for example:

$$\begin{aligned}\hat{a}_r &= \left(\hat{a}_r\cdot\hat{a}_x\right)\hat{a}_x + \left(\hat{a}_r\cdot\hat{a}_y\right)\hat{a}_y + \left(\hat{a}_r\cdot\hat{a}_z\right)\hat{a}_z \\ &= \sin\theta\cos\phi\ \hat{a}_x + \sin\theta\sin\phi\ \hat{a}_y + \cos\theta\ \hat{a}_z\end{aligned}$$

This result explicitly shows that $\hat{a}_r$ is a function of θ and ϕ.

For **example**, at the point in space $r = 7.239$, $\theta = 90^\circ$ and $\phi = 0^\circ$, we find that $\hat{a}_r = \hat{a}_x$. In other words, at this point in space, the direction $\hat{a}_r$ points in the ***x***-direction.

Or, at the point in space $r = 2.735$, $\theta = 90^\circ$ and $\phi = 90^\circ$, we find that $\hat{a}_r = \hat{a}_y$. In other words, at this point in space, $\hat{a}_r$ points in the ***y***-direction.

Additionally, we can write $\hat{a}_\theta$ and $\hat{a}_\phi$ as:

$$\hat{a}_\theta = (\hat{a}_\theta \cdot \hat{a}_x)\hat{a}_x + (\hat{a}_\theta \cdot \hat{a}_y)\hat{a}_y + (\hat{a}_\theta \cdot \hat{a}_z)\hat{a}_z$$

$$\hat{a}_\phi = (\hat{a}_\phi \cdot \hat{a}_x)\hat{a}_x + (\hat{a}_\phi \cdot \hat{a}_y)\hat{a}_y + (\hat{a}_\phi \cdot \hat{a}_z)\hat{a}_z$$

Alternatively, we can write **Cartesian** base vectors in terms of spherical base vectors, i.e.,

$$\hat{a}_x = (\hat{a}_x \cdot \hat{a}_r)\hat{a}_r + (\hat{a}_x \cdot \hat{a}_\theta)\hat{a}_\theta + (\hat{a}_x \cdot \hat{a}_\phi)\hat{a}_\phi$$

$$\hat{a}_y = (\hat{a}_y \cdot \hat{a}_r)\hat{a}_r + (\hat{a}_y \cdot \hat{a}_\theta)\hat{a}_\theta + (\hat{a}_y \cdot \hat{a}_\phi)\hat{a}_\phi$$

$$\hat{a}_z = (\hat{a}_z \cdot \hat{a}_r)\hat{a}_r + (\hat{a}_z \cdot \hat{a}_\theta)\hat{a}_\theta + (\hat{a}_z \cdot \hat{a}_\phi)\hat{a}_\phi$$

Using the **table** on the previous page, we can insert the result of each dot product to express each base vector in terms of **spherical coordinates**!